Egzamin dla Aktuariuszy z 28 maja 2012 r.

Matematyka Ubezpieczeń Majątkowych

Zadanie 1

OZN: Y to zmienna o dystrybuancie $F(x) = \frac{\sum q_i F_i(x)}{\sum q_i}$

Z_i to zmienna o dystrybuancie $F_i(x)$

$E(Z_i) = m_i$

$E(Z_i^2) = m_{2,i}$

$EY = \frac{\sum q_i m_i}{\sum q_i}, E(Y^2) = \frac{\sum q_i m_{2,i}}{\sum q_i}$

$\operatorname{var} X = \sum \operatorname{var}(X_i) = \sum q_i \left(m_{2,i} - q_i m_i^2\right)$

$\operatorname{var} Z = n\overline{q}\left[\frac{\sum q_i m_{2,i}}{\sum q_i} - \overline{q}\,\frac{\left(\sum q_i m_i\right)^2}{\left(\sum q_i\right)^2}\right]$

$ODP = \sum q_i m_{2,i} - \frac{1}{n}\left(\sum q_i m_i\right)^2 - \sum q_i m_{2,i} + \sum q_i^2 m_i^2 =$

$= \sum q_i^2 m_i^2 - \overline{mq}\sum q_i m_i = \sum_{i=1}^{n} q_i^2 m_i^2 - 2\overline{mq}\sum q_i m_i + \overline{mq}\sum q_i m_i =$

$= \sum_{i=1}^{n}\left[q_i^2 m_i^2 - 2\overline{mq}q_i m_i + \overline{mq}q_i m_i\right] = \sum_{i=1}^{n}\left[q_i m_i - \overline{mq}\right]^2$

Zadanie 2

Dla w=2 mamy:

$P(X = 0{,}5w) = P(X = 1) = 0{,}2$

$P(X = 0) = 0{,}8$

$\rightarrow EX = 0{,}2$

$P = (1+\theta)\cdot EX \cdot (1-\alpha) = 1{,}25 \cdot 0{,}2(1-\alpha) = 0{,}25(1-\alpha)$

Szukamy $E[u(w - P - \alpha X)] \rightarrow \max$

$u(w - P - \alpha X) = u(2 - 0{,}25(1-\alpha) - \alpha X)$

$Eu(w - P - \alpha X) = E\ln(2 - 0{,}25(1-\alpha) - \alpha X) = E\ln(1{,}75 + 0{,}25\alpha - \alpha X) =$

$= 0{,}2\ln(1{,}75 + 0{,}25\alpha - \alpha) + 0{,}8\ln(1{,}75 + 0{,}25\alpha) = 0{,}2\ln(1{,}75 - 0{,}75\alpha) + 0{,}8\ln(1{,}75 + 0{,}25\alpha) = f(\alpha)$

$f'(\alpha) = \frac{-0{,}2\cdot 0{,}75}{1{,}75 - 0{,}75\alpha} + \frac{0{,}8\cdot 0{,}25}{1{,}75 + 0{,}25\alpha} = 0$

$\frac{-0{,}15(1{,}75 + 0{,}25\alpha) + 0{,}2(1{,}75 - 0{,}75\alpha)}{(1{,}75 - 0{,}75\alpha)(1{,}75 + 0{,}25\alpha)} = 0 \rightarrow$

$-0{,}2625 - 0{,}0375\alpha + 0{,}35 - 0{,}15\alpha = 0$

$0{,}1875\alpha = 0{,}0875$

$$\alpha = \frac{7}{15}$$

Zadanie 3

Szukamy funkcji generującej momenty
$Y = X_1 + ... + X_N, gdzie \quad N \cong ujemny.dwumianowy(r,q)$

$$P(X_i = 1) = Q$$
$$P(X_i = 0) = P = 1 - Q$$
$$M_X(t) = e^t Q + P$$

$$M_Y(t) = M_N(\ln(M_X(t))) = \left[\frac{1-q}{1-q(e^t Q + P)}\right]^r = \left[\frac{\frac{1-q}{1-qP}}{1 - \frac{qQ}{1-qP}e^t}\right]^r$$

Sprawdzamy, że $1 - \frac{qQ}{1-qP} = \frac{1-q}{1-qP} = \frac{p}{1-qP}$

$\rightarrow Y \cong ujemny.dwumianowy\left(r; \frac{qQ}{1-qP}\right)$

Zadanie 4

$$\Psi(u) = \frac{e^{-Ru}}{1+\theta}, R = \frac{\beta\theta}{1+\theta} \rightarrow \Psi(u) = \frac{\exp\left(-\frac{\beta\theta}{1+\theta}\cdot u\right)}{1+\theta}$$

$$\frac{1}{10} = \frac{\exp\left(-\frac{0{,}2\beta}{1{,}2}u\right)}{1{,}2} \rightarrow \exp\left(-\frac{0{,}2\beta}{1{,}2}u\right) = 0{,}12$$

$$ODP = \frac{\exp\left(-2u\frac{0{,}2\beta}{1{,}2}\right)}{1{,}2} = \frac{\left[\exp\left(-\frac{0{,}2\beta}{1{,}2}u\right)\right]^2}{1{,}2} = \frac{0{,}12^2}{1{,}2} = 0{,}012$$

Zadanie 5

$$g(d_1) = g(d_2) + (d_2 - d_1)(1 - F(d_2)) + \left(E\left(X \middle| X \in (d_1, d_2]\right) - d_1\right)(F(d_2) - F(d_1))$$

Mamy:

$$ODP = g(10) = E[(X-20)_+] + 10(1 - F(20)) + \left(E\left(X \middle| 10 < X \le 20\right) - 10\right)(F(20) - F(10)) =$$

$$= 8 + 10 \cdot \left(1 - \frac{3}{4}\right) + (13 - 10) \cdot \left(\frac{3}{4} - \frac{1}{4}\right) = 10{,}5 + 1{,}5 = 12$$

Zadanie 6

$ODP = P(T_{n+1} + D_{n+1} < T_n + D_n) = P(T_{n+1} - T_n + D_{n+1} < D_n) = P(X + Z < Y)$ gdzie
$X \cong wykl(1); Z, Y \cong wykl(0{,}5)$ i niezależne

$$P(X+Z \le t) = \int_0^t \int_0^{t-x} 0{,}5e^{-0{,}5z} e^{-x} dzdx = \int_0^t e^{-x}\left(1 - e^{-0{,}5(t-x)}\right) =$$
$$= 1 - e^{-t} - \int_0^t e^{-0{,}5t} e^{-0{,}5x} dx = 1 - e^{-t} - e^{-0{,}5t}\left(2 - 2e^{-0{,}5t}\right) =$$
$$= 1 - e^{-t} - 2e^{-0{,}5t} + 2e^{-t} = 1 + e^{-t} - 2e^{-0{,}5t}$$
$$f_{X+Z}(t) = e^{-0{,}5t} - e^{-t}$$
$$ODP = \int_0^\infty \int_x^\infty 0{,}5e^{-0{,}5y}\left(e^{-0{,}5x} - e^{-x}\right) dydx = \int_0^\infty \left(e^{-0{,}5x} - e^{-x}\right) e^{-0{,}5x} dx = 1 - \frac{2}{3} = \frac{1}{3}$$

Zadanie 7

$$f_{-U(T)|T<\infty} = \frac{1}{p_1}\left[1 - P(y)\right] = \begin{cases} 1 & y < 1 \\ 0 & y \ge 1 \end{cases} \to J(0{,}1) \qquad P(y) = \begin{cases} 0 & y < 1 \\ 1 & y \ge 1 \end{cases} \quad p_1 = 1$$
$$E\left(e^{-U(T)R} \middle| T < \infty\right) = \int_0^1 e^{uR} du = \frac{e^R - 1}{R}$$
$$\Psi(u) = \frac{e^{-Ru} R}{e^R - 1} = \frac{\lambda}{c} e^{-Ru} \quad \text{bo } 1 + \theta = \frac{c}{\lambda EY}$$
$$M_Y(R) = 1 + (1+\theta)EY \cdot R \to a)\ e^R = 1 + \frac{cR}{\lambda} \to \frac{e^R - 1}{R} = \frac{c}{\lambda}$$
$$\left(\frac{1}{2}\right)^{u+1} \le \frac{\lambda}{c} e^{-Ru} \le \left(\frac{1}{2}\right)^u$$
$$\frac{1}{2}\left(\frac{e^R}{2}\right)^u \le \frac{\lambda}{c} \le \left(\frac{e^R}{2}\right)^u$$

Jeżeli $e^R > 2$ to obie strony dążą do nieskończoności i dla jakiegoś u nie będzie spełnione bo $\frac{\lambda}{c}$ ustalone stałe; tak samo jest dla $e^R < 2 \to e^R = 2$

Z a) i powyższego mamy: $2 = 1 + \frac{c \ln 2}{\lambda} \to c = \frac{\lambda}{\ln 2}$

Zadanie 8

n – ilość ryzyk

$$ODP = \sum_{i=1}^{n} P\left(Y > 9 \middle| \text{i szkod}\right) P\left(\text{i szkod} \middle| \text{co najmniej 1 szkoda}\right)$$

$$P(Y>9|\text{i szkod})=\sum_{j=1}^{i}P(Y_j>9)\frac{1}{i}$$

$$P(Y_j>9)=\int_0^\infty P(Y>9|\beta)f(\beta)=\int_0^\infty e^{-9\beta}\beta e^{-\beta}d\beta=0{,}01$$

$$P(Y>9|\text{i szkod})=\sum_{j=1}^{i}0{,}01\cdot\frac{1}{i}=0{,}01$$

$$P(\text{i szkod}|\text{co najmniej 1 szkoda})=\frac{\binom{n}{i}q^i(1-q)^{n-i}}{1-(1-q)^n}$$

$$ODP=\frac{\sum_{i=1}^{n}0{,}01\binom{n}{i}q^i(1-q)^{n-i}}{1-(1-q)^n}=\frac{0{,}01\left[1-(1-q)^n\right]}{1-(1-q)^n}=0{,}01$$

Zadanie 9

T_2-t ma rozkład wykładniczy bo brak pamięci
$t-T_1$ ma rozkład wykładniczy bo brak pamięci

$$\to E(T_2-T_1|T_2>t>T_1)=1+1=2$$

Zadanie 10

$$\operatorname{var}(S(n))=E(\operatorname{var}(S(n)|\theta))+\operatorname{var}(E(S(n)|\theta))$$
$$E(S(n))=EE(S(n)|\theta)$$
$$E(S(n)|\theta)=E(Y|\theta)\cdot n\lambda=\mu(\theta)n\lambda$$
$$E(S(n))=E(\mu(\theta)n\lambda)=\mu n\lambda$$
$$\operatorname{var}(S(n)|\theta)=n\lambda(\operatorname{var}(Y|\theta)+E^2(Y|\theta))=n\lambda\left[(\mu(\theta))^2+\mu^2(\theta)\right]$$
$$\operatorname{var}(S(n))=E\left[2n\lambda\mu^2(\theta)\right]+\operatorname{var}[n\lambda\mu(\theta)]=$$
$$=2n\lambda\left[\operatorname{var}(\mu(\theta))+E^2(\mu(\theta))\right]+n^2\lambda^2\operatorname{var}(\mu(\theta))=2n\lambda(a^2+\mu^2)+n^2\lambda^2a^2$$

$$\frac{\operatorname{var}(S(n))}{[E(S(n))]^2}=\frac{2n\lambda(a^2+\mu^2)+n^2\lambda^2a^2}{\mu^2n^2\lambda^2}:\frac{n^2\lambda^2}{n^2\lambda^2}=\frac{a^2+\frac{2}{n\lambda}(a^2+\mu^2)}{\mu^2}$$