Egzamin dla Aktuariuszy z 28 maja 2012 r.

Matematyka Ubezpieczeń na Życie

Zadanie 1

$${}_{E(T(x))}p_x = \exp\left(-\int_x^{x+E(T(x))}\mu_s ds\right) = c$$ liczymy obustronnie pochodną czyli:

$${}_{E(T(x))}p_x' = 0$$

$$f(x) = \exp\left(-\int_x^{x+E(T(x))}\mu_s ds\right) = \exp\left(-\int_0^{x+E(T(x))}\mu_s ds + \int_0^x \mu_s ds\right)$$

$$E(T(x)) = \int_0^\infty {}_s p_x ds$$

$$E'(T(x)) = \int_0^\infty {}_s p_x (\mu_x - \mu_{x+s}) ds = \mu_x E(T(x)) - 1$$

$$f'(x) = {}_{E(T(x))}p_x \cdot \left[-(x + E(T(x)))' \mu_{x+E(T(x))} + \mu_x\right] =$$

$$= {}_{E(T(x))}p_x \cdot \left[-\mu_{x+E(T(x))} \cdot (\mu_x E(T(x)) - 1 + 1) + \mu_x\right] =$$

$$= {}_{E(T(x))}p_x \left[\mu_x \left(1 - E(T(x))\mu_{x+E(T(x))}\right)\right] = 0$$

$$\to 1 - E(T(x))\mu_{x+E(T(x))} = 0 \to E(T(x)) = \frac{1}{\mu_{x+E(T(x))}}$$

Zadanie 2

n>m>0

$$\bar{A}_{x:\overline{n+1}|} = \bar{A}^1_{x:\overline{n+1}|} + v^{n+1}\,{}_{n+1}p_x = \bar{A}^1_{x:\overline{n}|} + v^{n+1}\,{}_n p_x q_{x+n} + v^{n+1}\,{}_{n+1}p_x =$$

$$= \bar{A}_{x:\overline{n}|} - v^n\,{}_n p_x + v^{n+1}\,{}_n p_x - v^{n+1}\,{}_n p_x p_{x+n} + v^{n+1}\,{}_{n+1}p_x =$$

$$= \bar{A}_{x:\overline{n}|} - v^n\,{}_n p_x (1 - v) = \bar{A}_{x:\overline{n}|} - {}_n E_x \left(1 - e^{-\delta}\right)$$

$$\bar{a}_{x:\overline{m+1}|} = \frac{1 - \bar{A}_{x:\overline{m+1}|}}{\delta} = \frac{1 - \bar{A}_{x:\overline{m}|} + {}_m E_x\left(1 - e^{-\delta}\right)}{\delta} = \frac{\delta\bar{a}_{x:\overline{m}|} + {}_m E_x\left(1 - e^{-\delta}\right)}{\delta}$$

$$\bar{P}(m+1, n+1) = \frac{\bar{A}_{x:\overline{n+1}|}}{\bar{a}_{x:\overline{m+1}|}} = \frac{\left(\bar{A}_{x:\overline{n}|} - {}_n E_x\left(1 - e^{-\delta}\right)\right)\delta}{\delta\bar{a}_{x:\overline{m}|} + {}_m E_x\left(1 - e^{-\delta}\right)} =$$

$$= \frac{0{,}0488 \cdot \left(0{,}188 - 0{,}093 \cdot \left(1 - e^{-0{,}0488}\right)\right)}{0{,}0488 \cdot 12{,}854 + 0{,}352 \cdot \left(1 - e^{-0{,}0488}\right)} \approx 0{,}0139$$

Zadanie 3

$$P_{x:\overline{25}|}\ddot{a}_{x+5:\overline{25}|} = (100000 - D)A^1_{x+5:\overline{25}|} + 100000 A_{x+5:\overline{25}|}^{\ \ 1}$$

$$D = \frac{100000 A_{x+5:\overline{25}|}^{\ \ 1} - P_{x:\overline{25}|}\ddot{a}_{x+5:\overline{25}|}}{A^1_{x+5:\overline{25}|}} + 100000 =$$

$$= \frac{100000 \cdot \left[v^{25}\ {}_{25}p_{x+5} - \frac{1-(1-v)\ddot{a}_{x:\overline{25}}}{\ddot{a}_{x:\overline{25}}}\ddot{a}_{x+5:\overline{25}} \right]}{1-(1-v)\ddot{a}_{x+5:\overline{25}} - v^{25}\ {}_{25}p_{x+5}} + 100000 =$$

$$= \frac{100000 \cdot \left[0{,}95^{25} \cdot 0{,}442 - \frac{1-0{,}05 \cdot 12{,}8}{12{,}8} \cdot 12{,}108 \right]}{1-0{,}05 \cdot 12{,}108 - 0{,}95^{25} \cdot 0{,}442} + 100000 \approx 19880$$

Zadanie 4

$$1000\int_0^\infty e^{-0,05t}e^{-0,04t}e^{-0,06t}dt = M\int_0^\infty e^{-0,05t}e^{-0,04t}e^{-0,06t} \cdot 0{,}04 + \int_0^\infty 1000te^{-0,05t}e^{-0,04t}e^{-0,06t} \cdot 0{,}06dt$$

$$1000\int_0^\infty e^{-0,15t}dt = M \cdot 0{,}04\int_0^\infty e^{-0,15t}dt + 60\int_0^\infty te^{-0,15t}dt$$

$$\frac{1000}{0{,}15} = \frac{0{,}04M}{0{,}15} + \frac{60}{0{,}15^2}$$

$$M = \frac{150-60}{0{,}15^2} \cdot \frac{0{,}15}{0{,}04} = \frac{90}{0{,}006} = 15000$$

Zadanie 5

$$Z = b\frac{1-e^{-\delta T}}{\delta} + ae^{-\delta T} = \frac{b}{\delta} + e^{-\delta T}\left(a - \frac{b}{\delta}\right)$$

$$\operatorname{var} Z = \left(a - \frac{b}{\delta}\right)^2 \operatorname{var}\left(e^{-\delta T}\right) = \left(a - \frac{b}{\delta}\right)^2 \left[{}^2\overline{A}_x - \left(\overline{A}_x\right)^2\right]$$

$$\frac{\operatorname{var}(3{,}1)}{\operatorname{var}(6;0{,}9)} = \frac{\left(3 - \frac{1}{\delta}\right)^2}{\left(6 - \frac{0{,}9}{\delta}\right)^2} = 1{,}4523$$

Ponieważ $0 < \delta < 0{,}1 \to 3 - \frac{1}{\delta} < 0$ i $6 - \frac{0{,}9}{\delta} < 0 \to \frac{\left|3 - \frac{1}{\delta}\right|}{\left|6 - \frac{0{,}9}{\delta}\right|} = \sqrt{1{,}4523}$

$$i\ \frac{\frac{1}{\delta} - 3}{\frac{0{,}9}{\delta} - 6} = \sqrt{1{,}4523}$$

$$1 - 3\delta = (0{,}9 - 6\delta)\sqrt{1{,}4523}$$

$$\delta\left(6\sqrt{1{,}4523} - 3\right) = 0{,}9\sqrt{1{,}4523} - 1$$

$$\delta = \frac{0{,}9\sqrt{1{,}4523} - 1}{6\sqrt{1{,}4523} - 3}$$

$$ODP=\frac{\mathrm{var}(8;0{,}8)}{\mathrm{var}(3{,}1)}=\frac{\left(3-\frac{0{,}8}{\delta}\right)^2}{\left(3-\frac{1}{\delta}\right)^2}\approx 0{,}46$$

Zadanie 6

$$\frac{\Pi^s(t)}{V(t)}=f=const\rightarrow\frac{V'(t)-\delta V(t)}{V(t)}=f\rightarrow\frac{V'(t)}{V(t)}=f+\delta=const\rightarrow V(t)=c_1e^{c_2t}$$

$$\begin{cases}V(t_1)=c_1e^{c_2t_1}=0{,}4\\ V(t_2)=c_1e^{c_2t_2}=0{,}9\end{cases}\rightarrow c_1=0{,}4e^{-c_2t_1}\ \text{ i wstawiamy do II}$$

$$0{,}4e^{-c_2t_1}e^{c_2t_2}=0{,}9$$

$$\exp(c_2(t_2-t_1))=\frac{9}{4}$$

$$c_2=\frac{\ln\left(\frac{9}{4}\right)}{t_2-t_1}\rightarrow c_1=0{,}4\exp\left(-\frac{t_1}{t_2-t_1}\ln\left(\frac{9}{4}\right)\right)$$

$$V(t)=0{,}4\cdot\left(\frac{9}{4}\right)^{-\frac{t_1}{t_2-t_1}}\left(\frac{9}{4}\right)^{\frac{t}{t_2-t_1}}=0{,}4\cdot\left(\frac{9}{4}\right)^{\frac{t-t_1}{t_2-t_1}}$$

$$V\left(\frac{t_1+t_2}{2}\right)=0{,}4\cdot\left(\frac{9}{4}\right)^{\frac{\frac{t_1+t_2}{2}-t_1}{t_2-t_1}}=0{,}4\cdot\left(\frac{9}{4}\right)^{\frac{t_2-t_1}{2(t_2-t_1)}}=\frac{4}{10}\left(\frac{9}{4}\right)^{0{,}5}=\frac{4}{10}\frac{3}{2}=0{,}6$$

Zadanie 7

B – koszty administracyjne
A – koszty początkowe
P – składka brutto

$$\begin{cases}(*)100000A_x+A+B\ddot{a}_x=P\ddot{a}_x\\ (**)A+B=P+2000\\ (***)B\ddot{a}_x=2A\end{cases}$$

$$ODP=1-\frac{N}{P},\ gdzie\ \ N=\frac{100000A_x}{\ddot{a}_x}$$

Z (***) $12{,}5B=2A\rightarrow A=6{,}25B$

Wstawiamy do (**)

$6{,}25B+B=P+2000\rightarrow P=7{,}25B-2000$ i wstawiamy do (*)

$$100000A_x+6{,}25B+12{,}5B=(7{,}25B-2000)\ddot{a}_x$$

$$\rightarrow B=\frac{100000A_x+2000\ddot{a}_x}{7{,}25\ddot{a}_x-18{,}75}=\frac{100000(1-\ddot{a}_x(1-v))+2000\ddot{a}_x}{7{,}25\ddot{a}_x-18{,}75}=$$

$$= \frac{100000 \cdot (1-0{,}05 \cdot 12{,}5) + 2000 \cdot 12{,}5}{7{,}25 \cdot 12{,}5 - 18{,}75} = \frac{62500}{71{,}876}$$

$$P = 7{,}25B - 2000 = \frac{453125}{71{,}876} - 2000$$

$$ODP = 1 - \frac{\frac{1-(1-v)\ddot{a}_x}{\ddot{a}_x} \cdot 100000}{P} = 1 - \frac{1-0{,}05 \cdot 12{,}5}{12{,}5P} \cdot 100000 \approx 30{,}3\%$$

Zadanie 8

$$E(0) \cdot \bar{a}_{\overline{xy}} = 1 \rightarrow E(0) = \frac{1}{\bar{a}_{\overline{xy}}}$$

$$E(40)A = 1 \rightarrow E(40) = \frac{1}{A}$$

$$A = \int_0^{40} e^{-0{,}03t} dt + e^{-40 \cdot 0{,}03} \left[{}_{40}p_x \, {}_{40}q_y \bar{a}_{x+40} + {}_{40}p_y \, {}_{40}q_x \bar{a}_{y+40} + {}_{40}p_x \, {}_{40}p_y \bar{a}_{x+40:y+40} \right]$$

$$ODP = \frac{\bar{a}_{\overline{xy}}}{A}$$

$$\bar{a}_x = \bar{a}_{x+40} = \int_0^{\infty} e^{-0{,}03t} e^{-0{,}01t} dt = 25$$

$$\bar{a}_y = \bar{a}_{y+40} = \int_0^{\infty} e^{-0{,}03t} e^{-0{,}01t} dt = 20$$

$$\bar{a}_{\overline{x+40:y+40}} = \bar{a}_{\overline{xy}} = \bar{a}_x + \bar{a}_y - \bar{a}_{xy} = 25 + 20 - \int_0^{\infty} e^{-0{,}03t} e^{-0{,}01t} e^{-0{,}02t} = 25 + 20 - \frac{50}{3} = \frac{85}{3}$$

$$A = \frac{1-e^{-1{,}2}}{0{,}03} + e^{-1{,}2} \left[e^{-0{,}4}\left(1-e^{-0{,}8}\right) \cdot 25 + e^{-0{,}8}\left(1-e^{-0{,}4}\right) \cdot 20 + \frac{85}{3} e^{-1{,}2} \right] =$$

$$= \frac{100}{3}\left(1-e^{-1{,}2}\right) + 25e^{-1{,}6}\left(1-e^{-0{,}8}\right) + 20e^{-2}\left(1-e^{-0{,}4}\right) + \frac{85}{3} e^{-2{,}4} \rightarrow ODP = 0{,}96$$

Zadanie 9

$$ODP = v^{\frac{2}{3}} \, {}_{\frac{2}{3}}p_{60}^{(w_1)} \left[\frac{1}{2} \, {}_{\frac{1}{3}}p_{60\frac{2}{3}}^{(w_2)} \cdot v^{\frac{1}{3}} \ddot{a}_{61}^{(w_2)} \right] \cdot 3600$$

$${}_{\frac{2}{3}}p_{60}^{(w_1)} = \frac{70 - 60 - \frac{2}{3}}{70-60} = \frac{28}{30} = \frac{14}{15}$$

$${}_{\frac{1}{3}}p_{60\frac{2}{3}}^{(w_2)} = \frac{90 - 60\frac{2}{3} - \frac{1}{3}}{90 - 60\frac{2}{3}} = \frac{29}{88} \cdot 3 = \frac{87}{88}$$

$$\ddot{a}^{(w_2)}_{61} = \sum_{k=0}^{29} v^k \, {}_k p^{(w_2)}_{61} = \sum_{k=0}^{29} v^k \, \frac{29-k}{29}$$

$$X = v + 2v^2 + ... + 29v^{29}$$
$$Xv = v^2 + 2v^3 + ... + 29v^{30}$$
$$X(1-v) = v + v^2 + ... + v^{29} - 29v^{30} = \frac{1-v^{29}}{i} - 29v^{30}$$
$$\rightarrow X = \frac{1-v^{29}}{i(1-v)} - \frac{29v^{30}}{1-v}$$

$$\ddot{a}^{(w_2)}_{61} = \frac{1-v^{30}}{1-v} - \frac{1}{29}\left(\frac{1-v^{29}}{i(1-v)} - \frac{29v^{30}}{1-v}\right)$$
$$ODP = 36000 \cdot 0,95 \cdot \frac{14}{15}\frac{1}{2}\frac{87}{88}\left[\frac{1-0,95^{30}}{0,05} - \frac{1}{29}\left(\frac{1-0,95^{29}}{0,05 \cdot \frac{5}{95}} - \frac{29 \cdot 0,95^{30}}{0,05}\right)\right] \approx 155500$$

Zadanie 10

a=25
r=65
s(r)=0,7
$n(t) = 100e^{0,04t}$

$$\overline{a}^{h}_{65} = \int_{65}^{\infty} e^{-\delta(x-65)} \frac{s(x)}{s(65)} dx = |x-65=t| = \int_{0}^{\infty} e^{-\delta t} \, {}_t p_{65} dt = \overline{a}_{65} = 15$$
$$P(60) = 0,7 \cdot 12000 \overline{a}_{65} \int_{25}^{65} e^{-\delta(65-x)} \cdot 100e^{0,04(60-x+25)} m(x) dx =$$
$$= 8400 \cdot 15 \int_{25}^{65} e^{-0,04(65-x)} \cdot 100e^{0,04 \cdot 85} e^{-0,04x} m(x) dx =$$
$$= 8400 \cdot 15 \cdot 100e^{0,04 \cdot 85} e^{-0,04 \cdot 65} \underbrace{\int_{25}^{65} m(x) dx}_{=1} = 8400 \cdot 15 \cdot 100e^{0,8} \approx 28041820$$