Egzamin dla Aktuariuszy z 28 maja 2012 r.

Prawdopodobieństwo i Statystyka

Zadanie 1

$$P(\ln X < t) = P(X < e^t) = \int_1^{e^t} \frac{1}{\theta} x^{-\frac{1}{\theta}-1} = \left[-x^{-\frac{1}{\theta}} \right]_1^{e^t} = 1 - e^{-\frac{t}{\theta}} \cong wykl\left(\frac{1}{\theta}\right)$$

$$P(\min(\ln X_1,...,\ln X_n) < t) = 1 - P(\min(\ln X_1,...,\ln X_n) > t) = 1 - e^{-\frac{tn}{\theta}} \cong wykl\left(\frac{n}{\theta}\right)$$

$$E(T_n) = E(aY) = aEY = a \cdot \frac{\theta}{n} \to a = n \text{ i } T_n = nY$$

Zbiór $\left| nY - \frac{1}{3} \right| > \frac{1}{6} \to \begin{Bmatrix} nY - \frac{1}{2} > 0 \text{ dla } Y > \frac{1}{3n} \\ nY < \frac{1}{6} \text{ dla } Y < \frac{1}{3n} \end{Bmatrix} = \left\{ Y > \frac{1}{2n} \wedge Y < \frac{1}{6n} \right\}$

$$\to P\left(\left| nY - \frac{1}{3} \right| > \frac{1}{6} \right) = P\left(Y < \frac{1}{6n} \right) + P\left(Y > \frac{1}{2n} \right) =$$

$$= 1 - e^{-\frac{n}{\theta}\cdot\frac{1}{6n}} + e^{-\frac{n}{\theta}\frac{1}{2n}} = 1 - e^{-\frac{1}{2}} + e^{-\frac{3}{2}} = 1 + \exp\left(-\frac{3}{2}\right) - \exp\left(-\frac{1}{2}\right)$$

Zadanie 2

$$\text{var}\left(\frac{S_N}{N}\right) = E\left[\text{var}\left(\frac{S_N}{N}|\theta\right)\right] + \text{var}\left[E\left(\frac{S_N}{N}|\theta\right)\right]$$

$$\text{var}\left(\frac{S_N}{N}|\theta\right) = E\,\text{var}\left(\frac{S_N}{N}|N\right) + \text{var}\,E\left(\frac{S_N}{N}|N\right) = E\left(\frac{1}{N^2}\cdot 100\theta^2 N\right) + \text{var}\left(\frac{1}{N}\cdot 100N\right) =$$

$$= E\left[\frac{100\theta^2}{N}|\theta\right] + \text{var}(10\theta|\theta) = 100\theta^2 E\left(\frac{1}{N}|\theta\right)$$

$$E\left(\frac{S_N}{N}|\theta\right) = EE\left(\frac{S_N}{N}|N\right) = E\left(\frac{1}{N}\cdot 10\theta N\right) = E(10\theta|\theta) = 10\theta$$

$$E\left(\frac{1}{N}|\theta\right) = \sum_{i=1}^{\infty} \frac{1}{i}\cdot i \cdot (1-\theta)^{i-1}\theta^2 = \sum_{i=1}^{\infty}(1-\theta)^{i-1}\theta^2 = \frac{\theta^2}{1-\theta}\frac{1-\theta}{\theta} = \theta$$

$$\text{var}\left(\frac{S_N}{N}|\theta\right) = 100\theta^2 \cdot \theta = 100\theta^3$$

$$\text{var}\left(\frac{S_N}{N}\right) = E(100\theta^3) + \text{var}(10\theta) = 100E\theta^3 + 100\,\text{var}\,\theta$$

$$E\theta^3 = \int_0^1 6\theta^4(1-\theta) = \left[\frac{6\theta^5}{5} - \frac{6\theta^6}{6}\right]_0^1 = \frac{6}{5} - 1 = 0,2$$

$$E\theta^2 = \int_0^1 6\theta^3(1-\theta) = \left[\frac{6\theta^4}{4} - \frac{6\theta^5}{5}\right]_0^1 = \frac{3}{2} - \frac{6}{5} = 0{,}3$$

$$E\theta = \int_0^1 6\theta^2(1-\theta) = \left[\frac{6\theta^3}{3} - \frac{6\theta^4}{4}\right]_0^1 = 2 - \frac{3}{2} = 0{,}5$$

$$\operatorname{var}\theta = 0{,}3 - 0{,}5^2 = 0{,}05$$

$$\operatorname{var}\left(\frac{S_N}{N}\right) = 100 \cdot 0{,}2 + 100 \cdot 0{,}05 = 25$$

Zadanie 3

$$L = \left(\frac{1}{\sqrt{2\Pi}\sigma}\right)^2 e^{-\frac{(y_1-b)^2}{2\sigma^2}} e^{-\frac{(y_2-b)^2}{2\sigma^2}} e^{-\frac{\left(y_3-b\sqrt{5}\right)^2}{2\sigma^2}} e^{-\frac{(y_4-3b)^2}{2\sigma^2}} e^{-\frac{(y_5-3b)^2}{2\sigma^2}}$$

$$\ln L = -5\ln\sqrt{2\Pi} - 5\ln\sigma - \frac{(y_1-b)^2 + (y_2-b)^2 + \left(y_3 - b\sqrt{5}\right)^2 + (y_4-3b)^2 + (y_5-3b)^2}{2\sigma^2}$$

$$\frac{\partial \ln L}{\partial b} = \frac{2(y_1-b) + 2(y_2-b) + 2\sqrt{5}\left(y_3 - b\sqrt{5}\right) + 6(y_4-3b) + 6(y_5-3b)}{\sigma^3} = 0$$

$$\to y_1 - b + y_2 - b + \sqrt{5}y_3 - 5b + 3y_4 - 9b + 3y_5 - 9b = 0$$

$$\to \hat{b} = \frac{y_1 + y_2 + \sqrt{5}y_3 + 3y_4 + 3y_5}{25}$$

$$\frac{-5\sigma^2 + (y_1-b)^2 + (y_2-b)^2 + \left(y_3 - b\sqrt{5}\right)^2 + (y_4-3b)^2 + (y_5-3b)^2}{\sigma^3} = 0$$

$$\to \hat{\sigma} = \sqrt{\frac{\left(y_1-\hat{b}\right)^2 + \left(y_2-\hat{b}\right)^2 + \left(y_3-\hat{b}\sqrt{5}\right)^2 + \left(y_4-3\hat{b}\right)^2 + \left(y_5-3\hat{b}\right)^2}{5}}$$

$$y_i \cong N\left(0;\sigma^2\right)$$

$$\left(y_1-\hat{b}\right)^2 + \left(y_2-\hat{b}\right)^2 + \left(y_3-\hat{b}\sqrt{5}\right)^2 + \left(y_4-3\hat{b}\right)^2 + \left(y_5-3\hat{b}\right)^2 =$$

$$= y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 - 2y_1\hat{b} - 2y_2\hat{b} - 2\sqrt{5}y_3\hat{b} - 6y_4\hat{b} - 6y_5\hat{b} + \hat{b}^2 + \hat{b}^2 + 5\hat{b}^2 + 9\hat{b}^2 + 9\hat{b}^2 =$$

$$= \sum_{i=1}^{5} y_i^2 - 2\hat{b}\left(y_1 + y_2 + \sqrt{5}y_3 + 3y_4 + 3y_5\right) + 25\hat{b}^2 = \sum_{i=1}^{5} y_i^2 - 25\hat{b}^2$$

Analogicznie do standardowego przypadku dowodzimy, że $\frac{\hat{b}}{\hat{\sigma}}$ ma rozkład t-Studenta

$$\hat{b} \cong N\left(0;\frac{\sigma^2}{25}\right) \to \frac{5\hat{b}}{\sigma} \cong N(0{,}1) \qquad \frac{\sum y_i}{\sigma^2} \cong \chi^2(5) \to$$

$$\frac{\sum y_i^2}{\sigma^2} - \frac{\left(5\hat{b}\right)^2}{\sigma^2} \cong \chi^2(4) \quad \text{bo} \quad \left(\frac{5\hat{b}}{\sigma}\right)^2 \cong \chi^2(1)$$

$$\frac{\hat{b}}{\hat{\sigma}} = \frac{\frac{5\hat{b}}{\sigma}}{2 \cdot \sqrt{\frac{\sum y_i^2 - \left(5\hat{b}\right)^2}{4\sigma^2}} \cdot \frac{1}{\sqrt{5}}\sigma} = \frac{\sqrt{5}}{10} X \quad \text{gdzie } X \cong t(4)$$

$$P\left(\left|\frac{\hat{b}}{\hat{\sigma}} > c\right|\right) = P\left(|X| > \frac{10}{\sqrt{5}} c\right) = 0{,}05$$

Z tablic: $\frac{10}{\sqrt{5}} c = 2{,}776 \rightarrow c \approx 0{,}62$

Zadanie 4

$$\int_r \frac{\left(\frac{1}{r} - a\right)^2}{\frac{1}{r^2}} f\left(r \middle| x_1,\dots,x_n\right) dr \rightarrow \min$$

$$f\left(r \middle| x_1,\dots,x_n\right) = \frac{f\left(x_1,\dots,x_n \middle| r\right) f(r)}{f\left(x_1,\dots,x_n\right)}$$

$$f\left(x_1,\dots,x_n\right) = \int_0^\infty \left(\frac{\sqrt{r}}{\sqrt{2\Pi}}\right)^n e^{-\frac{r}{2}\sum x_i^2} \frac{\beta^\alpha}{\Gamma(\alpha)} r^{\alpha-1} e^{-\beta r} =$$

$$= \int_0^\infty \frac{1}{\left(\sqrt{2\Pi}\right)^n} \frac{\beta^\alpha}{\Gamma(\alpha)} r^{\alpha-1+\frac{n}{2}} e^{-r\left(\frac{\sum x_i^2}{2}+\beta\right)} = \left| \begin{array}{l} \alpha := \alpha + \frac{n}{2} \\ \beta := \beta + \frac{\sum x_i^2}{2} \end{array} \right| = \frac{1}{\left(\sqrt{2\Pi}\right)^n} \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma\left(\alpha + \frac{n}{2}\right)}{\left(\beta + \frac{\sum x_i^2}{2}\right)^{\alpha+\frac{n}{2}}}$$

$$f\left(r \middle| x_1,\dots,x_n\right) = \frac{\left(\frac{\sqrt{r}}{\sqrt{2\Pi}}\right)^n e^{-\frac{r}{2}\sum x_i^2} \frac{\beta^\alpha}{\Gamma(\alpha)} r^{\alpha-1} e^{-\beta r}}{\frac{1}{\left(\sqrt{2\Pi}\right)^n} \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\beta + \frac{\sum x_i^2}{2}\right)^{-\alpha-\frac{n}{2}} \Gamma\left(\alpha + \frac{n}{2}\right)} \cong \Gamma\left(\alpha + \frac{n}{2}; \beta + \frac{\sum x_i^2}{2}\right)$$

$$\int_r \frac{\left(\frac{1}{r} - a\right)^2}{\frac{1}{r^2}} f\left(r \middle| x_1,\dots,x_n\right) = \int_r (1-ar)^2 f\left(r \middle| x_1,\dots,x_n\right) = \int_r \left(1 - 2ar + a^2 r^2\right) f\left(r \middle| x_1,\dots,x_n\right) =$$

$$= 1 - a \cdot 2 \underbrace{\frac{\alpha + \frac{n}{2}}{\beta + \frac{\sum x_i^2}{2}}}_{B} + a^2 \cdot \underbrace{\frac{\left(\alpha + \frac{n}{2}\right)\left(\alpha + 1 + \frac{n}{2}\right)}{\left(\beta + \frac{\sum x_i^2}{2}\right)^2}}_{A} = f(a)$$

$$a_{\min} = -\frac{B}{2A} = \frac{2\left(\alpha+\frac{n}{2}\right)}{\beta+\frac{\sum x_i^2}{2}} \frac{\left(\beta+\frac{\sum x_i^2}{2}\right)^2}{2\left(\alpha+\frac{n}{2}\right)\left(\alpha+1+\frac{n}{2}\right)} = \frac{\beta+\frac{1}{2}\sum_{i=1}^{n} x_i^2}{\frac{n}{2}+\alpha+1}$$

Zadanie 5

$ODP = P_{H_0}(K)$

Ponieważ dystrybuanty są ciągłe to prawdopodobieństwo, że dwa elementy w próbie się powtórzą jest zerowe więc przy H_0 $\overline{\Omega} = (3+5)! = 8!$ - tyle jest wszystkich możliwości (elementy X_i, Y_i ustawione według wielkości) prawdopodobieństwo każdego układu jest równe $\frac{1}{\overline{\Omega}}$

Możliwe rangi x-ów spełniające warunki:

Dla S<10
(1,2,3) (1,2,4) (1,2,5) (1,2,6) (1,3,4) (1,3,5) (2,3,4) – 7

Dla S>17
(8,7,6) (8,7,5) (8,7,4) (8,7,3) (8,6,5) (8,6,4) (7,6,5) – 7

Czyli jest 7+7=14 możliwości

Ilość układów gdy rangi x-ów ustalone wynosi 3!5!

$$ODP = \frac{14 \cdot 3! \cdot 5!}{8!} = \frac{14 \cdot 6}{6 \cdot 7 \cdot 8} = \frac{2}{8} = 0{,}250$$

Zadanie 6

$S_5 \cong N(5;20)$

$S_{20} \cong N(20;80)$

$\operatorname{cov}(S_5, S_{20}) = \operatorname{cov}(S_5; S_5 + S_{6:20}) = \operatorname{var} S_5 = 20$

Z regresji:

$$E(X|Y=y) = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)$$

$$D^2(X|Y=y) = (1-\rho^2)\sigma_1^2$$

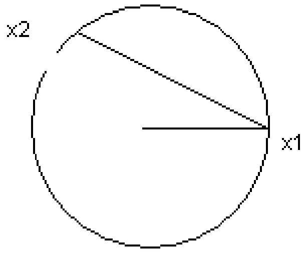
$$E(S_5|S_{20}) = 5 + \frac{20}{80}(S_{20}-20) = 0{,}25S_{20}$$

$$\operatorname{var}(S_5|S_{20}) = \left(1 - \frac{20^2}{20 \cdot 80}\right) \cdot 20 = 15$$

$$E(S_5^2|S_{20} = 24) = 15 + (0{,}25 \cdot 24)^2 = 51$$

Zadanie 7

$$a = 2r\sin\frac{\varphi}{2}$$



r=1
x1 – ustalone
$\varphi \cong J(0;2\Pi)$

$$a = \begin{cases} 2r\sin\frac{\varphi}{2} & \varphi \in (0;\Pi) \\ 2r\sin\frac{2\Pi-\varphi}{2} & \varphi \in (\Pi;2\Pi) \end{cases}$$

$$ODP = \int_0^{\Pi} \frac{1}{2\Pi} 2\sin\frac{\varphi}{2} d\varphi + \int_{\Pi}^{2\Pi} \frac{1}{2\Pi} 2\sin\frac{2\Pi-\varphi}{2} d\varphi = |2\Pi - \varphi = x| =$$

$$= \frac{1}{\Pi}\left[-2\cos\frac{\varphi}{2}\right]_0^{\Pi} + \int_0^{\Pi} \frac{1}{\Pi}\sin\frac{x}{2} = \frac{2}{\Pi} + \left[\frac{1}{\Pi}\left(-2\cos\frac{x}{2}\right)\right]_0^{\Pi} = \frac{2}{\Pi} + \frac{2}{\Pi} = \frac{4}{\Pi}$$

Zadanie 8

OZN: $T = \max(|X_1|,|X_2|,|X_3|)$

$I.\ dla\ \ \theta > 1$

$$\frac{p_\theta(x_1,x_2,x_3)}{p_1(x_1,x_2,x_3)} = \begin{cases} \left(\frac{1}{\theta}\right)^3 & T < 1 \\ \infty & T > 1 \end{cases}$$

II. dla $\theta < 1$

$$\frac{p_\theta(x_1,x_2,x_3)}{p_1(x_1,x_2,x_3)} = \begin{cases} \left(\frac{1}{\theta}\right)^3 & T < \theta \\ 0 & T > \theta \end{cases}$$

I. szukamy testu najmocniejszego dla $H_1 : \theta_1\ \ \theta_1 > 1$ ustalone
A – podzbiór taki, że $T \in (0,x)$ $\mathrm{T} < 1$ i $\mathrm{P}_1(A) = \alpha \rightarrow Z = A \cup [1;\infty)$ jest zbiorem krytycznym testu najmocniejszego (spełnia warunek wystarczający dla testu najmocniejszego w lemacie Neymana-Pearsona)

Moc testu jest równa:

$$\beta = P_{\theta_1}(A) + P_{\theta_1}(T > 1)$$

$$P_{\theta_1}(A) = \frac{\alpha}{\theta^3}$$

$$P_{\theta_1}(T > 1) = 1 - P_{\theta_1}(T < 1) = 1 - \frac{1}{\theta^3}$$

$$\beta = 1 - \frac{1-\alpha}{\theta^3}$$

Rodzina rozkładów jednostajnych jest dla statystyki T rodziną z monotonicznym ilorazem wiarygodności stąd K_1 dla $H_0 : \theta < 1$ przeciw $H_1 : \theta > 1$ jest postaci $K_1 = \{T > c\}$ gdzie c:

$$P_1(K_1) = 1 - P^3(|X| < c) = 1 - c^3 = \alpha \to c = \sqrt[3]{1-\alpha}$$

$$moc = P_{\theta_1}(K_1) = 1 - \left(\int_{-c}^{c} \frac{1}{2\theta_1}\right)^3 = 1 - \frac{c^3}{\theta_1^3} = 1 - \frac{1-\alpha}{\theta^3}$$

Czyli wśród testów najmocniejszych dla testu $H_0 : \theta = 1$ przeciw $H_1 : \theta = \theta_1$ jest test JNM $\to K_1 = \{T \in (0;c) \cup T > 1\}$ gdzie $P_1(T \in (0,c)) = \alpha$ jest testem JNM na poziomie istotności α

II. dla weryfikacji $H_0 : \theta = 1$ wobec hipotezy alternatywnej $H_1 : \theta < 1$ otrzymujemy że test o obszarze krytycznym $\{T \in (0,e) \cup T > 1\}$ gdzie e spełnia $P_1(T < e) = \alpha$ jest testem JNM na poziomie istotności α (test spełnia warunek wystarczający lematu Neymana-Pearsona dla każdej alternatywy postaci $H_1 : \theta = \theta_1$ gdzie $\theta_1 < 1$ i nie zależy od wyboru θ_1

Łączymy I i II:

$dla\ c = \sqrt[3]{\alpha}$ zbiory krytyczne pokrywają się

$$\to K = \{T < \sqrt[3]{0,2}\} \cup \{T > 1\}$$

$$ODP = P(K) = P\{T < \sqrt[3]{0,2}\} + P\{T > 1\}$$

$$P\{T < \sqrt[3]{0,2}\} = P^3(|X| < \sqrt[3]{0,2}) = \left[-\int_{-\sqrt[3]{0,2}}^{0} \frac{1}{4}x + \int_{0}^{\sqrt[3]{0,2}} \frac{1}{4}x\right]_0^3 = 0,000625$$

$$P\{T > 1\} = 1 - P^3(|X| < 1) = 1 - \left[-\int_{-1}^{0} \frac{1}{4}x + \int_{0}^{1} \frac{1}{4}x\right]^3 = 1 - \frac{1}{64} = \frac{63}{64}$$

$$ODP = \frac{63}{64} + 0,000625 = 0,985$$

Zadanie 9

$dla\ \ i \in \{4,5,6,7,8,9,10\}$

$$P(N_{X1} = i | N_2 = 5) = \frac{P(N_{X1} = i \wedge N_2 = 5)}{P(N_2 = 5)}$$

$$P\left(N_{X1}=i\wedge N_2=5\right)=\frac{\binom{14}{5}\binom{10}{i}\binom{6}{10-i}}{\binom{30}{15}}$$

$$P\left(N_2=5\right)=\frac{\binom{14}{5}\binom{16}{10}}{\binom{30}{15}}$$

$$P\left(N_{X1}=i\middle|N_2=5\right)=\frac{\binom{14}{5}\binom{10}{i}\binom{6}{10-i}}{\binom{14}{5}\binom{16}{10}}=\frac{\binom{10}{i}\binom{6}{10-i}}{\binom{16}{10}}$$

Weźmy zmienną $X=N_{X1}-4$ dla której $\operatorname{var}\left(X\middle|N_2=5\right)=\operatorname{var}\left(N_{X1}\middle|N_2=5\right)$

Wtedy dla $k\in\{0,1,...,6\}$

$$P\left(X=k\middle|N_2=5\right)=\frac{\binom{10}{k+4}\binom{6}{6-k}}{\binom{16}{10}}=\frac{\binom{6}{k}\binom{10}{6-k}}{\binom{16}{10}}\cong hipergeometryczny(N,M,n)$$

Gdzie N=16,M=10,n=6

$$\to ODP=npq\frac{N-n}{N-1},gdzie\ \ p=1-q=\frac{M}{N}$$

$$ODP=6\cdot\frac{10}{16}\cdot\frac{6}{16}\cdot\frac{16-6}{15}=\frac{15}{16}$$

Zadanie 10

$$\operatorname{cov}(U,X_0)=E\operatorname{cov}\left(U,X_0\middle|X_0\right)+\operatorname{cov}\left(E\left(U\middle|X_0\right),E\left(X_0\middle|X_0\right)\right)=\operatorname{cov}\left(E\left(U\middle|X_0\right),X_0\right)$$

$E\left(U\middle|X_0\right)=\int_{X_0}^{\infty}X_0f(x)+\int_0^{X_0}xf(x)$ gdzie f(x) to gęstość $\min\left(X_1,...X_n\right)$

$$P\left(\min\left(X_1,...,X_n\right)<t\right)=1-P\left(\min\left(X_1,...X_n\right)>t\right)=1-\left(\frac{1}{1+t}\right)^{3n}\ \to\min\left(X_1,...,X_n\right)\cong Pareto(3n,1)$$

$$E\left(U\middle|X_0\right)=\frac{x_0}{(1+x_0)^{3n}}+3n\left[-\frac{1}{(3n-1)t^{3n-1}}+\frac{1}{3nt^{3n}}\right]_1^{1+x_0}=\frac{x_0}{(1+x_0)^{3n}}-\frac{3n}{(3n-1)(1+x_0)^{3n-1}}+\frac{1}{(1+x_0)^{3n}}+\frac{3n}{3n-1}-1$$

$$=\frac{1}{3n-1}\left[1-\frac{1}{(1+x_0)^{3n-1}}\right]$$

$$\operatorname{cov}(U,X_0)=\operatorname{cov}\left(X_0,E\left(U\middle|X_0\right)\right)=-\frac{1}{3n-1}\operatorname{cov}\left(X_0,\frac{1}{(1+X_0)^{3n-1}}\right)$$

$$E\left(\frac{x_0}{\left(1+x_0\right)^{3n-1}}\right)=\int_0^{\infty}\frac{3x}{(1+x)^{3n+3}}=\frac{3}{3n+1}-\frac{3}{3n+2}$$

$$E\left(\frac{1}{\left(1+x_0\right)^{3n-1}}\right)=\int_0^{\infty}\frac{3}{(1+x)^{3n+3}}=\frac{3}{2(3n+2)}$$

$$\operatorname{cov}\left(X_0,\frac{1}{\left(1+X_0\right)^{3n-1}}\right)=\frac{6(3n+2)-6(3n+1)-3(3n+1)}{2(3n+1)(3n+2)}=\frac{-3(3n-1)}{2(3n+1)(3n+2)}$$

$$\operatorname{cov}\left(U,X_0\right)=\frac{1}{3n-1}\frac{3(3n-1)}{2(3n+1)(3n+2)}=\frac{3}{2(3n+1)(3n+2)}$$